Symmetric Functions and Macdonald Polynomials Seminar Outline

Spring 2023

- If a proof uses material from a different lecture than yours, please remind the audience what the corresponding statement is by writing it out on the blackboard.
- If you don't understand notation(we use a lot of references) or have any questions, please email me ccl2166@columbia.edu even if we have already met before your talk.
- If you find a proof hard to understand or find it annoying, you can instead give examples demonstrating the theorem or lemma instead of presenting the proof.
- Material in parenthesis are optional and should only be covered if there's time. Also "until Theorem A" means that you end by covering Theorem A.

1 Symmetric Functions

(1) **Generating Functions**

  (a) Follow [1, Section 1, 2, 4].
    - For Section 2, only cover Section 2.5, the Product Rule.

  (b) Follow [8, Section 1, 3, 4].
    - Skip Section 3 part (5).

(2) **Monomial and Elementary Symmetric Functions**

  (a) Follow [5, Chapter 1.2].

  (b) Follow [5, Chapter 2.1].
    - For Example 2.7, prove Proposition 2.10 first and then go back and write e_{31} as a linear combination of monomial symmetric functions.
    - Skip Page 37 and go directly to Proposition 2.18 instead.

(3) **Complete Homogeneous and Power Sum Symmetric Functions**

  (a) Follow [5, Chapter 2.2].
    (a) Skip combinatorial proof of 2.24, Corollary 2.26.

  (b) Follow [5, Chapter 2.3].
    - Skip Corollary 2.32.

  (c) Cover Exercise 2.17 (a) and Exercise 2.27 in [5] (Ask Cailan for notes).

(4) $q-$**analogs**

  (a) Define $q-$numbers (Equation (3.2) on page 75) and $q-$analogues and then prove that the generating function for inversions is a $q-$analog for $n!$ [12, Theorem 3.2.1].

  (b) Start on page 77 of [12] and go until Theorem 3.2.4.

(c) Cover [4, Section 6.4].

(5) **Stirling Numbers and Evaluation of Symmetric Functions**

(a) Cover [5, Proposition 3.1] in Chapter 3.1 only.

(b) Cover [5, Chapter 3.2].

- Skip Proposition 3.5, 3.6.

(c) Cover [5, Chapter 3.3].

- Skip Proposition 3.17.

(d) Cover [5, Chapter 3.4] starting at Proposition 3.27.

(6) **Schur Polynomials**

(a) Cover [5, Chapter 4.1] up to Definition 4.11.

- Skip to page 86 and state Proposition 4.15.
- Start again on page 87 and follow to the end of the section.

(b) Cover [5, Chapter 4.2] up to Proposition 4.28 on page 96.

(7) **Jacobi-Trudi identities**

(a) State [5, Theorem 6.2] and then show the examples on page 158, 159 starting with "When we study our answer..." State [5, Theorem 6.10] and show the examples on the top of page 172.

(b) Follow the proof of Jacobi-Trudi from [3, Proposition 4.2].

(c) Follow [5, Chapter 6.3] up to the proof of Proposition 6.20.

- Prove Eq (3.4) before proving proposition 6.19.
- Apply ω to the first Jacobi-Trudi identity to get the second.

(8) **The Robinson-Schensted (RS) Algorithm**

(a) Follow [13, Chapter 3.1-3.3].

(b) Follow [13, Chapter 3.5] up to the statement of Theorem 3.5.3.

(c) State [13, Theorem 3.6.6].

(9) **The Hall Inner Product**

(a) Follow [5, Chapter 7.1]

- End on the definition of $\delta_{\lambda\mu}$ on page 191 and start again in the middle of page 193 at , "There is a tool..."
- Just state Proposition 7.3.

(b) Follow [5, Chapter 7.2]

(c) Follow [5, Chapter 7.3].

- Just state Proposition 7.17.

(10) **The Robinson-Schensted-Knuth (RSK) Algorithm and Cauchy's Formula**

(a) Follow [14, Section 7.1] ending with the statement of Theorem 7.11.5.

(b) Follow [14, Chapter 7.2]

- Stop after the end of the proof of Theorem 7.12.1.
- Cover Corollary 7.12.5 on page 324.

(c) State [14, Theorem 7.13.1]. Present your favorite corollaries from 7.13.6 to 7.13.9 on page 330.

(11) **Pieri and Murnaghan–Nakayama Rules**

(a) Cover [5, Chapter 9.1].

- Stop at the end of page 251 and then do Example 9.5. Prove Theorem 9.7 and do examples.

(b) Cover [5, Chapter 9.2].

- Stop in the middle of page 259 where it says, "is a horizontal strip of length $n - k$."
- State Theorem 9.17 and then continue on page 266 near the top where it says, "We can apply..."

(12) **The Hook-length Formula and the Littlewood-Richardson Rule**

(a) Let f^λ be the number of semistandard Young tableaux of shape λ. Cover [12, Chapter 3.10] ending on the top of page 125 after listing all possible tableaux.

(b) Prove the Hook-length formula following [2, Chapter 4.6].

- The formula for $F_\mu(N)$ on page 82 is special in it's own right. It's called the Hook-Content Formula and gives a formula for the number of semistandard young tableaux of shape μ with fillings from the set $\{1, 2, \ldots, N\}$. Do more examples after the proof of Hook-length.

(c) The Littlewood-Richardson coefficients $c^\lambda_{\mu\,\nu}$ are defined to be the coefficients

$$s_\mu \cdot s_\nu = \sum_\lambda c^\lambda_{\mu\,\nu} s_\lambda$$

Now follow [11, pages 3-8].

- Skip anything involving $s_{\lambda\backslash\mu}$.
- Skip S_3 symmetry on page 3.

(13) **HyperGeometric Series**

(a) Follow [9] ending before Theorem 6 on page 6.

(14) **Plane Partitions**

(a) Define Plane Partitions following the beginning of the corresponding wikipedia article ending with, "This formula may be viewed as the 2-dimensional analogue of Euler's product formula for the number of integer partitions of n."

(b) Define $\mathcal{B}(r, s, t)$ following [3, page 13] and state [3, Theorem 1.3].

(c) Follow [3, page 130] starting at "Using Schur Functions" until page 133.

(d) Prove that

$$\sum_{\pi\in\mathcal{B}(r,s,t)} q^{|\pi|} = \prod_{i=1}^{r}\prod_{j=1}^{t}\frac{1-q^{i+j+s-1}}{1-q^{i+j-1}} = \prod_{i=1}^{r}\prod_{j=1}^{t}\prod_{j=1}^{s}\frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}}$$

and now by letting $r, s, t \to \infty$ above, show that

$$\sum_{n=0}^{\infty} \mathrm{PL}(n)x^n = \prod_{\ell=1}^{\infty}\frac{1}{(1-x^\ell)^\ell}$$

(Ask Cailan for notes)

2 Macdonald Polynomials

(15) **The Original Macdonald Polynomials**

(a) Cover [10, Chapter 1.11]. Eq (9.3) in [2] gives an example.

- Ignore anything mentioning zonal, Jack, Hall-Littlewood polynomials.
- Stop in the middle of page 15 after P_λ satisfy (11.1) and (11.2).

(b) Cover [10, Chapter 1.12].

- Stop at the end of Eq (12.6)

(16) **Combinatorial formula for** $H_\mu(z_n; q, t)$

(a) Follow [2, Chapter 9.5] ending at the top of page 157 where it says, "as $s_3+(q+t)s_{21}+qts_{111}$"

- Ignore the part on the bottom of page 156 on quasisymmetric monomial basis and monomial symmetric functions.
- Illustrate the theorem of Haglund, Haiman, Loehr Eq (9.23) by computing H_3 and/or H_{111} and checking your answers in [2, page 205].

(b) State Proposition 9.2 in [7]. Do some computations when $\mu = (2, 2)$ for some λ and check your answer by looking at H_{22} in [2, page 205].

(17) **Macdonald Constant Term Conjecture and** $q-$**Hypergeometric Series**

(a) Follow [15] pages 5-11 and page 25-26.

(b) Follow [6] Sections 1-12.

- Skip Section 3.
- Section 6 is optional as we have seen these theorems multiple times already.
- Equation (7.8) is Euler's Pentagonal number theorem and $3n^2 + n/2$ are pentagonal numbers (they form pentagons).
- Skip Sections 8-11.
- (Section 17 is famous if you have time for it).

Also DO NOT omit q from $F(a, b; t : q)$.

(18) **Plethysm and the Modified Macdonald Polynomials**

References

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[15] O. Warnaar. Macdonald polynomials made easy. https://people.smp.uq.edu.au/OleWarnaar/talks/Macdonald.pdf.